Research on the recurrence relations for the spin-weighted spheroidal harmonics

Guihua Tian[1,2] Wang Hui-hui*
1.School of Science, Beijing University of Posts And Telecommunications. Beijing 100876, China. and
2.State Key Laboratory of Information Photonics and Optical Communications,
Beijing University of Posts And Telecommunications. Beijing 100876, China.
(Dated:)

The spin-weighted spheroidal harmonics (SWSHs) are important for the study of the perturbation of the Kerr blackhole, which is relevant with the most tests in relativity. SWSHs with the spin-weight $s=0$ are the spheroidal functions and are easier to study than the counterparts of the spin-weight $s\neq 0$ Through the methods in super-symmetric quantum mechanics, we give the recurrence relations for SWSHs with different spin weight. These relations can be applied to derive SWSHs with $s=\pm 1,\ \pm 2$ from the spheroidal functions.. They also give SWSHs of $s=-\frac{1}{2},\pm\frac{3}{2}$ from that of $s=\frac{1}{2}$. These recurrence relations are first investiged and are very important both in theoretical background and the astrophysical applications. `Keywords`: spin-weighted spheroidal harmonics , recurrence relation, super-symmetric quantum mechanics, shape-invariance

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I. INTRODUCTION

The spin-weighted spheroidal harmonics were

$$\frac{1}{\sin\theta}\frac{dS}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right)+\left[s+a^2\omega^2\cos^2\theta\right.$$
$$\left.+2a\omega s\cos\theta-\frac{(m+s\cos\theta)^2}{\sin^2\theta}+E\right]S=0,\qquad(1)$$

and first defined by Teukolsky in the context of the perturbation to the Kerr black holes. where a represent the angular momentum per unit mass of the rotating black hole and ω is the frequency of the perturbation fields. The parameter s, the spin-weight of the perturbation fields, could be $s=0,\pm\frac{1}{2},\pm 1,\pm 2$, and Ψ corresponds to the scalar, neutrino, electromagnetic or gravitational perturbations respectively.

For simplicity, we will put the parameters $a\omega=\beta$ in the following. Eq.(1) is also called spin-weighted spheroidal wave equation. Together with the boundary condition of $S(\theta)$ finite at $\theta=0,\ \pi$, Eq.(1) constituted a kind of the singular Sturm-Liuvelle eigenvalue problem. The eigenfunctions $S_n(\theta), n=0,1...$ to the Sturm-Liuvelle problem are called the spin-weighted spheroidal harmonics (SWSHs). Actually, $S_n(\theta)$ also depends on the parameters m and s, and should be denoted as $S_n(\theta,m,s)$[33]. Note we will denote the parameters m, s in S_n through their appearance in the super-potentials W, and see the following for details.

SWSHs could help to deepen our understanding of many astrophysical processes modeled as stable problems in Kerr black hole (BH) [1]-[20]. With real frequency, they are used to separate the angular dependence of the gravitational radiation produced by perturbation to the Kerr BH. As angular basis, SWSHs in Kerr BH are indispensable in all physical problems connecting with the perturbation. A variety of physical situations need to use SWSHs, including the astrophysical problems involving the study of quasi-normal modes (QNMs) of Kerr black hole, quantum field theory in curved space-time and studies of D-branes, etc [6]-[20]. For instance, an important application of SWSHs concerning an astrophysical problem is the determination of black hole parameters from gravitational wave observations, like to determine M, a, the source location and the black hole's spin orientation from the observed waveform. An investigation of all these issues require the calculation of "scalar products" between different quasinormal modes, and, in particular, between the SWSHs describing their angular dependence.

SWSHs are also necessary for computing the characteristic resonances of Kerr black holes, which will involve the complex parameter. In the framework of semi-classical general relativity, it has been conjectured that the highly damped resonances may shed light on the quantum properties of black holes[16]-[19]. For rotating black holes, these highly damped resonances are characterized by the imaginary part of the frequency approaching infinity. Therefore, the solutions (SWSHs) are important in the theoretical background and have attracted considerable attentions all the time[1]-[20]. Further detailed study on SWSHs is still very important.

We have investigated SWSH equations in low frequency cases by the use of the super-symmetric quantum mechanics methods (SUSYQM) and obtained the SWSHs by their recurrence relations [21]-[30].

Usually, the spheroidal harmonics (as the $s=0$ case for SWSHs) were thoroughly investigated in past [2]. SWSHs for $s\neq 0$ only appeared after 1970s and Eq.(1) is no longer invariant under the transformation of $\theta\to\pi-\theta$ whenever $s\neq 0$, which might be the cause of the more complex calculation involved in Eq.(1) and result in the lack of uniform conclusion concerning SWSHs's WKB approximation[9],[3]-[20]. All of these stimulate us to

*Electronic address: tgh-2000@263.net, wendy8151@163.com

seek the solutions of Eq.(1) for the case $s \neq 0$ in an alternative way, which has been used in the study of spherical harmonics.

From Eq.(1), it is easy to see that SWSHs reduce to the spin-weighted spherical harmonics when $\beta = 0$ and $S_n(\theta, m, s)e^{im\phi}$ are generally written as ${}_sY_{l,m}$ with $l = m + n$. Generally, the spherical Harmonics ${}_0Y_{l,m} = P_l^m(\cos(\theta))e^{im\phi}$ is in the contents of college level and are easier to grasp, while ${}_sY_{l,m}$ for $s \neq 0$ are not so easy. Nevertheless, one could obtain ${}_sY_{l,m}$ through recurrence relations as [3]

$$\begin{aligned} {}_{s-1}Y_{lm} &= A_{lm}[\frac{d}{d\theta} + \frac{m + scos\theta}{\sin\theta}]_sY_{lm}, && (2)\\ {}_{s+1}Y_{lm} &= B_{lm}[\frac{d}{d\theta} - \frac{m + scos\theta}{\sin\theta}]_sY_{lm}, && (3)\\ A_{lm} &= [(l+s)(l-s+1)]^{-\frac{1}{2}}, \\ B_{lm} &= [(l-s)(l+s+1)]^{-\frac{1}{2}}, \\ m &= 0, 1, 2, \ldots, l = m, m+1, \cdots. \end{aligned}$$

So, there also are the two ways to obtain ${}_sY_{lm}$, either by solving the corresponding differential equations or utilizing the recurrence relations Eqs.(2)-(3) and ${}_0Y_{lm} = P_l^m(\cos\theta)$. Actually, the recurrence relations of Eqs.(2)-(3) are very important in many situation . In the flat space-time, they will provide a method to obtain the electromagnetic field contents from the scalar field.

Similarly, the extension of Eqs.(2)-(3) to SWSHs is the same important as that in flat case, and will make one obtain the physical insight of electromagnetic and gravitational perturbation to Kerr black hole from the information of the scalar perturbation field. So they are worthy of efforts to study and are the main topics in the paper. The rest of this paper is divided into five sections. After introduction and review of the SUSYQM and spin-weighted spheroidal harmonics in section 2 and 3. We re-derive the recurrence relations for the spin-weighted spherical harmonics (SWSHs on the condition of $\beta = 0$) in section 4. In section 5, we extend the study of section 4 to the spin-weighted spheroidal harmonics with $\beta \neq 0$. and some conclusion will be given in the final section.

II. THE BRIEF INTRODUCTION OF THE SUPER-SYMMETRIC QUANTUM MECHANICS

In this section, we give a brief introduction of SUSYQM, which is powerful in solving the Schrödinger equation [32]:

$$-\frac{d^2\psi}{d\theta^2} + [V^-(x) - E]\psi = 0. \qquad (4)$$

A remark on the terms is perhaps necessary. In this paper, we will use the language of SUSYQM, of which the same nomenclatures are used mainly from their mathematical similarity with a Schrödinger equation in quantum mechanics (QM). We make a few remarks: Through Eq.(4) has the Schrödinger form, it need not to represent a real QM. Nevertheless, we will apply the nomenclatures in QM, such as the potential energy, the ground energy and state (sometimes ground eigen-vualue and eigenfunction), the excited energies and states, etc in the following for the sake of convenience.

In SUSYQM, it is mainly to factorize the Hamiltonian H^- of Eq.(4)

$$H^- = -\frac{d^2}{dx^2} + V^-(x) + E_0, \qquad (5)$$

as

$$H^- = \mathcal{A}^\dagger\mathcal{A}^-, \qquad (6)$$

where operators $\mathcal{A}^\dagger$, $\mathcal{A}^-$ are defined by the super-potential W as

$$\mathcal{A}^-(x) = \frac{d}{dx} + W(x), \quad \mathcal{A}^\dagger(x) = -\frac{d}{dx} + W(x). \qquad (7)$$

Eqs.(5)-(7) give the relations of the potential V and the super potential W as

$$W^2(x) - W'(x) = V^-(x) + E_0, \qquad (8)$$

which shows the superpotential W is completely determined by the potential V^- and E_0, where E_0 is the ground energy of Eq.(4). The superpotential W is focus in SUSYQM, it is connected with the ground function ψ_0 of Eq.(4) by

$$\begin{aligned} W(x) &= -\frac{\psi_0'(x)}{\psi_0(x)}, or && (9)\\ \psi_0 &= N\exp\Big[-\int W dx\Big]. && (10) \end{aligned}$$

where $'$ represent the derivative to x. Interchange the order of the operators $\mathcal{A}^\dagger$, $\mathcal{A}^-$ in Eq.(6) will give the partner Hamiltonian H^+ of H^-:

$$\begin{aligned} H^+ &= \mathcal{A}^-\mathcal{A}^\dagger = -\frac{d^2}{dx^2} + V^+ \\ &= -\frac{d^2}{dx^2} + W^2(x) + W'(x), && (11) \end{aligned}$$

where the super-potential W is connected with the potential V^+

$$V^+(x) = W^2(x) + W'(x). \qquad (12)$$

The partner Hamiltonians H^-, H^+ shares the same eigen-energies except the ground energy E_0 [32], and their eigenfunctions ψ_n^-, ψ_n^+ , that is

$$H^-\psi_n^- = (E_n - E_0)\psi_n^-, \; n = 0, 1, 2, \cdots \qquad (13)$$
$$H^+\psi_n^+ = (E_n - E_0)\psi_n^+, \; n = 1, 2, \cdots \qquad (14)$$

are related by

$$\psi_n^+ = \mathcal{A}^-\psi_n^-, \quad \psi_n^- = \mathcal{A}^\dagger\psi_n^+. \qquad (15)$$

The ground eigenfunction of H^- is given directly by the super-potential W. However, the excited state functions or eigen-functions cannot be obtained directly by the super-potential W, and needs some other properties of W, the shape invariance property. In order to introduce the shape-invariance concept, the super-potential W will be denoted by $W(x, a_1)$ with a_1 representing some parameters in W. So the pair of SUSY partner potentials $V^{\mp}(x, a_1)$ correspondingly become

$$V^{\mp}(x, a_1) = W^2(x, a_1) \mp W'(x, a_1), \tag{16}$$

and the corresponding partner Hamiltonians are

$$H_1^-(x; a_1) = -\frac{d^2}{dx^2} + V^-(x; a_1) = \mathcal{A}^\dagger(x; a_1)\mathcal{A}^-(x; a_1), \tag{17}$$

$$H_1^+(x; a_1) = -\frac{d^2}{dx^2} + V^+(x; a_1) = \mathcal{A}^-(x; a_1)\mathcal{A}^\dagger(x; a_1). \tag{18}$$

If the pair of partner potentials $V^\pm$ are similar in shape and different only from parameters, that is

$$V^+(x; a_1) = V^-(x; a_2) + R(a_1), \tag{19}$$

where $a_2 = f(a_1)$ and the remainder $R(a_1)$ is independent of x, then the super-potential W is said to be of shape-invariance. Through $a_2 = f(a_1)$ in the super-potential $W(x, a_2)$, one will have the new partner Hamiltonians:

$$H_2^-(x; a_2) = -\frac{d^2}{dx^2} + V^-(x; a_2) = \mathcal{A}^\dagger(x; a_2)\mathcal{A}^-(x; a_2), \tag{20}$$

$$H_2^+(x; a_2) = -\frac{d^2}{dx^2} + V^+(x; a_2) = \mathcal{A}^-(x; a_2)\mathcal{A}^\dagger(x; a_2). \tag{21}$$

The shape-invariance properties will result in the relations of the eigen-energies and eigen-functions of $H_2^-(x; a_2)$, $H_1^+(x; a_1)$ and make the Hamiltonian $H_1^-(x; a_1)$ completely integrable [32]. That is,

$$H_1^-(x; a_1) = -\frac{d^2}{dx^2} + V^-(x; a_1) = \mathcal{A}^\dagger(x; a_1)\mathcal{A}^-(x; a_1), \tag{22}$$

$$\begin{aligned} H_2^-(x; a_2) &= -\frac{d^2}{dx^2} + V^-(x; a_2) = \mathcal{A}^\dagger(x; a_2)\mathcal{A}^-(x; a_2) \\ &= H_1^+(x; a_1) - R(a_1) = \mathcal{A}^-(x; a_1)\mathcal{A}^\dagger(x; a_1) - R(a_1) \end{aligned} \tag{23}$$

meets the relation

$$\begin{aligned} H_2^-(x; a_2) &= \mathcal{A}^\dagger(x; a_2)\mathcal{A}^-(x; a_2) \\ = H_1^+(x; a_1) - R(a_1) &= \mathcal{A}^-(x; a_1)\mathcal{A}^\dagger(x; a_1) - R(a_1) \end{aligned} \tag{24}$$

with their wave functions $\psi_n^-(x; a_1), \psi_n^-(x; a_2)$ satisfying respectively

$$H_1^-(x; a_1)\psi_n^-(x; a_1) = E_n^1\psi_n^-(x; a_1), \tag{25}$$

$$H_2^-(x; a_2)\psi_n^-(x; a_2) = E_n^2\psi_n^-(x; a_2). \tag{26}$$

$\psi_n^-(x; a_1)$,$\psi_n^-(x; a_2)$ can be connected by the pair of operator $\mathcal{A}^-(x; a_1)$, $\mathcal{A}^\dagger(x; a_1)$ as [32]

$$\psi_n^-(x; a_1) = \mathcal{A}^\dagger(x; a_1)\psi_n^-(x; a_2) \tag{27}$$

$$\psi_n^-(x; a_2) = \mathcal{A}^-(x; a_1)\psi_n^-(x; a_1) \tag{28}$$

and $E_n^1 = E_n^2 + R(a_1)$. The Hamiltonians $H^-(x, a_1)$, $H^-(x, a_2)$ are the hierarchy of Hamiltonians we construct. Similarly, further Hamiltonian can be built by the further shape-invariance of the super potential $W(x, a_2)$, as we will use in the following. It is obvious that Eq.(27) and Eq.(28) are the recurrence relations, which will be applied to study SWSHs.

III. REVIEW OF THE SPIN-WEIGHTED SPHEROIDAL HARMONICS

In order to apply SUSYQM to Eq.(1), one should use the transformation [21]-[23]

$$S(\theta) = \frac{\psi(\theta)}{\sqrt{sin\theta}}. \tag{29}$$

to make it into a Schrödinger form

$$\begin{aligned} &\frac{d^2\psi}{d\theta^2} + \left[\frac{1}{4} + s + \beta^2\cos^2\theta - 2s\beta\cos\theta \right. \\ &\left. -\frac{(m + s\cos\theta)^2 - \frac{1}{4}}{\sin^2\theta} + E\right]\psi = 0. \end{aligned} \tag{30}$$

The super-potential W is expanded as the series sum of the parameter β as

$$W = W_0 + \sum_{n=1}^{\infty} \beta^n W_n. \tag{31}$$

In Refs.[21]-[27], the authors have studied the Eq.(30) with the general formula for W_n:

$$W_n(\theta) = \sum_{k=1}^{[\frac{n}{2}]} a_{n,k} \sin^{2k-1}\theta\cos\theta + \sum_{k=1}^{[\frac{n+1}{2}]} b_{n,k}\sin^{2k-1}\theta. \tag{32}$$

The first several expressions of W_n have been obtained in Refs.[26]-[30], of which we give the first three ones as follows:

$$\begin{aligned} W_0(\theta) &= \frac{b_{0,0} + a_{0,0}\cos\theta}{\sin\theta} = -\frac{s + (m+\frac{1}{2})\cos\theta}{\sin\theta}, &(33)\\ W_1(\theta) &= b_{1,1}\sin\theta = -\frac{s}{m+1}\sin\theta, &(34)\\ W_2(\theta) &= b_{2,1}sin\theta + a_{2,1}sin\theta\cos\theta, \\ &= -\frac{(m+s+1)(m-s+1)s}{(m+1)^3(2m+3)}sin\theta, \\ &+ \frac{(m+s+1)(m-s+1)}{(m+1)^2(2m+3)}sin\theta\cos\theta. &(35) \end{aligned}$$

The rest coefficients $a_{n,j}, b_{n,j}$ of W_n are described in detail in Refs.[26]-[30].

IV. THE RECURRENCE RELATIONS FOR THE SPIN-WEIGHTED SPHERICAL HARMONICS

As previously stated, SWSHs become to the spin-weighted spherical harmonics under special condition of

$\beta = 0$. In this section, we study the recurrence relations about the different spin-s fields for the spin-weighted spherical harmonics by the methods in SUSYQM, and verify that the results thus obtained are consistent with Eq.(2) previously obtained in Ref.[3].

We construct the idea of the hierarchy of Hamiltonians[32]:

$$H = H_1,\, H_2,\, H_3,\, \ldots$$

to study the kind of recurrence relations with the same parameter m but different s. here we choose the first three Hamiltonians H_1, H_2, H_3 to obtain the relations among them.

A. The useful formulas from the shape-invariance of the potentials

The first three Hamiltonians H_1, H_2, H_3 corresponding to the Hamiltonian in section 2 are rewritten as $H_1(\theta; a_1), H_2(\theta; a_2), H_3(\theta; a_3)$, whose corresponding potentials have the shape invariance property. In above, we put $a_1 = (A_0, B_0), a_2 = (C_0, D_0), a_3 = (E_0, F_0)$, where $A_0, B_0, C_0, D_0, E_0, F_0$ are constants.

Due to $\beta = 0$, the super potential becomes $W = W_0$. The super potentials corresponding to the first three Hamiltonian $H_1(\theta; A_0, B_0), H_2(\theta; C_0, D_0), H_3(\theta; E_0, F_0)$ are expressed by

$$W_0(\theta, A_0, B_0) = \frac{B_0 + A_0 \cos\theta}{\sin\theta}, \tag{36}$$

$$W_0(\theta, C_0, D_0) = \frac{D_0 + C_0 \cos\theta}{\sin\theta}, \tag{37}$$

$$W_0(\theta, E_0, F_0) = \frac{F_0 + E_0 \cos\theta}{\sin\theta}. \tag{38}$$

The definition of the shape-invariance of the potential gives the following

$$V_1^+(\theta; A_0, B_0) = V_1^-(\theta; C_0, D_0) + R(A_0, B_0), \tag{39}$$

$$V_2^+(\theta; C_0, D_0) = V_2^-(\theta; E_0, F_0) + R(C_0, D_0). \tag{40}$$

The above equations will provide the relations among the undetermined constants $A_0, B_0, C_0, D_0, E_0, F_0$, which turn out to be four ones corresponding to the four cases. The first is

$$C_0 = A_0 - 1, \quad D_0 = B_0, \tag{41}$$

$$E_0 = C_0 - 1 = A_0 - 2, \quad F_0 = D_0 = B_0, \tag{42}$$

and has been applied to obtain the spin-weighted spheroidal functions $S_n(\theta, m, s)$, $n > m$ from $S_n(\theta, m, s)$, $n = m$ in [27]-[28]. The second one is

$$C_0 = -A_0, \quad D_0 = -B_0, \tag{43}$$

$$E_0 = -C_0 = A_0, \tag{44}$$

$$F_0 = -D_0 = B_0, \tag{45}$$

and is trivial one. The last two are

$$C_0 = -B_0 - \frac{1}{2}, \quad D_0 = -A_0 + \frac{1}{2}, \tag{46}$$

$$E_0 = -D_0 - \frac{1}{2} = A_0 - 1, \tag{47}$$

$$F_0 = -C_0 + \frac{1}{2} = B_0 + 1, \tag{48}$$

and

$$C_0 = B_0 - \frac{1}{2}, \quad D_0 = A_0 - \frac{1}{2}, \tag{49}$$

$$E_0 = D_0 - \frac{1}{2} = A_0 - 1, \tag{50}$$

$$F_0 = C_0 + \frac{1}{2} = B_0 - 1, \tag{51}$$

which will be used to obtain the recurrence relations in the following. Some contents need to be noted here. The relations (46)-(51) are different from that in Refs.[27]-[28], and are important for the study of the recurrence relations of spin-weighted spheroidal harmonics with different spin s.

B. The recurrence relation from ${}_sY_{l,m+1}$ to ${}_{s-1}Y_{l,m+1}$

From $A_0 = -(m + \frac{1}{2})$, $B_0 = -s$ and Eqs.(33), (46)-(48), it is easy to obtain

$$W_0(\theta; A_0, B_0) = -\frac{s + (m + \frac{1}{2}) \cos\theta}{\sin\theta}, \tag{52}$$

$$W_0(\theta; C_0, D_0) = \frac{(m+1) + (s - \frac{1}{2}) \cos\theta}{\sin\theta}, \tag{53}$$

$$W_0(\theta; E_0, F_0) = -\frac{(s-1) + (m + \frac{3}{2}) \cos\theta}{\sin\theta}. \tag{54}$$

To use Eq.(27)-(28) repeatedly will produce the following recurrence relation

$$\psi_n(\theta; E_0, F_0) = \mathcal{A}^-(\theta; C_0, D_0)\mathcal{A}^-(\theta; A_0, B_0)\psi_n(\theta; A_0, B_0), \tag{55}$$

Then one defines

$$\psi_n(\theta; \widetilde{C_0}, \widetilde{D_0}) = \mathcal{A}^-(\theta; A_0, B_0)\psi_n(\theta; A_0, B_0) \tag{56}$$

with $\widetilde{C_0} = -(m + \frac{3}{2}), \widetilde{D_0} = -s$, which have been obtained in Refs.[27]-[28]. The Hamiltonian for $\psi_n(\theta; \widetilde{C_0}, \widetilde{D_0})$ is

$$H^-(\theta; \widetilde{C_0}, \widetilde{D_0}) = \mathcal{A}^\dagger(\theta; \widetilde{C_0}, \widetilde{D_0})\mathcal{A}^-(\theta; \widetilde{C_0}, \widetilde{D_0}). \tag{57}$$

The spin-weighted spherical harmonics $S(\theta)$ with the condition of spin s and magnetic quantum number $m+1$ are usually denoted by ${}_sY_{l,m+1}$. Following this tradition, we see that Eq.(29) is

$${}_{s-1}Y_{l,m+1} = \frac{\psi_n(\theta; E_0, F_0)}{\sqrt{sin\theta}}, \tag{58}$$

$${}_sY_{l,m+1} = \frac{\psi_n(\theta; \widetilde{C_0}, \widetilde{D_0})}{\sqrt{sin\theta}} \tag{59}$$

with $l = m+1+n$. So it is easy to observe that Eq.(55) about the eigenfunctions ψ_n is the recurrence relation of ${}_sY_{l,m}$ with the condition of spin $s \to s-1$ and the same parameter $m+1$.

By Eqs.(7), (52)-(55), (58)-(59), we have

$$\left[\frac{d}{d\theta} + \frac{(m+1)+scos\theta}{\sin\theta}\right]{}_sY_{l,m+1} = {}_{s-1}Y_{l,m+1}, \quad (60)$$

which is exactly the recurrence relations of ${}_sY_{l,m+1} \to {}_{s-1}Y_{l,m+1}$. which satisfy the condition of same m but different spin s.

With the parameter m changing into $m+1$ and ignoring the normalized constant, it is easy to see that Eq.(60) is of the same form as that in Eq.(2). Hence the recurrence relation Eq.(60) for SWSHs under the condition $\beta = 0$ is consistent with recurrence relations for the spin-weighted spherical harmonics from Ref.[3].

C. The recurrence relation from ${}_sY_{l,m+1}$ to ${}_{s+1}Y_{l,m+1}$

Similarly to the last subsection, we will use the relations (49)-(51) to obtain the recurrence (3). With

$$A_0 = -(m+\frac{1}{2}), B_0 = -s,$$

we see that

$$C_0 = -(s+\frac{1}{2}),\ D_0 = -(m+1), \quad (61)$$

$$E_0 = -(m+1+\frac{1}{2}),\ F_0 = -s-1. \quad (62)$$

The Hamiltonians $\mathcal{A}^+(\theta;A_0,B_0)\mathcal{A}^-(\theta;A_0,B_0)$ and $\mathcal{A}^+(\theta;E_0,F_0)\mathcal{A}^-(\theta;E_0,F_0)$ correspond to the transformed ones of spin-weighted spherical harmonics with m,s and $m+1, s+1$ respectively. Define $\psi_n(\theta;A_0,B_0),\ n = 0,1,\cdots$ are the eigen-functions for the Hamiltonian $\mathcal{A}^+(\theta;A_0,B_0)\mathcal{A}^-(\theta;A_0,B_0)$, then

$$\psi_n(\theta;E_0,F_0) = \mathcal{A}^-(\theta;C_0,D_0)\mathcal{A}^-(\theta;A_0,B_0)\psi_n(\theta;A_0,B_0), \quad (63)$$

are the eigenfunctions for or the Hamiltonian $\mathcal{A}^+(\theta;E_0,F_0)\mathcal{A}^-(\theta;E_0,F_0)$. So

$${}_{s+1}Y_{l,m+1} = \frac{1}{\sqrt{sin\theta}}\psi_n(\theta;E_0,F_0) \quad (64)$$

with $l = m+1+n$. With Eqs.(56), (59), we have

$${}_{s+1}Y_{l,m+1} = \frac{1}{\sqrt{sin\theta}}\mathcal{A}^-(\theta;C_0,D_0)\left[\sqrt{sin\theta}\,{}_sY_{l,m+1}\right] \quad (65)$$

with $l = m+1+n$. Eq.(65) turns out to be

$${}_{s+1}Y_{l,m+1} = \left[\frac{d}{d\theta} - \frac{(m+1)+scos\theta}{\sin\theta}\right]{}_sY_{l,m+1}, \quad (66)$$

So we obtain the recurrence relations Eqs.(60), (66) for the spin-weighted spherical harmonics,which are consistent with Eqs.(2)-(3), and we will extend the methods to the spin-weighted spheroidal harmonics with $\beta \neq 0$ in the following.

V. THE RECURRENCE RELATIONS FOR THE SWSHS UNDER THE CONDITION OF $\beta \neq 0$ I

In this part, we study the recurrence relations for SWSHs under the common condition of $\beta \neq 0$ by the methods in SUSYQM. There three parts in the section. Part A involves the introduction of the parameters into the super-potential. Part B and C are the extension of Eqs.(60), (66) to the case of $\beta \neq 0$.

A. The parameters for introduction of the shape-invariance potential

In order to make use of the shape-invariance properties of the hierarchy of Hamiltonians in SUSYQM, in Ref.[27] the authors introduce some constant parameters $A_{i,j} = 1,\ B_{i,j} = 1$ into the general formula of super-potential W . For the sake of simplicity, we denote the collection of $A_{i,j}, B_{i,j},\ i \le \frac{n}{2}+1, j \le n$ by $\mathcal{A}_n, \mathcal{B}_n$ and all the collection of $\mathcal{A}_n, \mathcal{B}_n,\ n=0,\ n=1,\cdots$ by $\mathcal{A}, \mathcal{B}$

$$W(\theta;\mathcal{A},\mathcal{B}) = W_0(\theta;\mathcal{A}_0,\mathcal{B}_0) + \sum_{n=1}^{\infty}\beta^n W_n(\theta;\mathcal{A}_n,\mathcal{B}_n), \quad (67)$$

where

$$W_0(\theta;\mathcal{A}_0,\mathcal{B}_0) = \frac{-A_{0,0}(m+\frac{1}{2})\cos\theta - sB_{0,0}}{\sin\theta}, \quad (68)$$

$$W_n(\theta;\mathcal{A}_n,\mathcal{B}_n) = \sum_{j=1}^{[\frac{n+1}{2}]}\bar{b}_{n,j}\sin^{2j-1}\theta + \cos\theta\sum_{j=1}^{[\frac{n}{2}]}\bar{a}_{n,j}\sin^{2j-1}\theta, \quad (69)$$

with

$$\bar{a}_{n,j} = A_{n,j}a_{n,j},\ \bar{b}_{n,j} = B_{n,j}b_{n,j}. \quad (70)$$

Some of $a_{n,j}, b_{n,j}$ for $n \le 2$ are given in Eqs.(33)-(35). For the other terms of $a_{n,j}, b_{n,j}$, see Refs.[26]-[30]. By the same way, We also represent the collection $C_{i,j}, D_{i,j}, \frac{n}{2}+1, j \le n$ by $\mathcal{C}_n, \mathcal{D}_n$ and $E_{i,j}, F_{i,j}, \frac{n}{2}+1, j \le n$ by $\mathcal{E}_n, \mathcal{F}_n$. Introducing parameters $\mathcal{C}, \mathcal{D}$ to be the set of $\mathcal{C}_n, \mathcal{D}_n,\ n = 0,1,2,\cdots$ and $\mathcal{E}, \mathcal{F}$ to be the set of $\mathcal{E}_n, \mathcal{F}_n,\ n = 0,1,2,\cdots$.

The super potential can be written as $W(\theta;\mathcal{C},\mathcal{D})$. Then

$$W(\theta;\mathcal{C},\mathcal{D}) = W_0(\theta;C_{0,0},D_{0,0}) + \sum_{n=1}^{\infty}\beta^n W_n(\theta;\mathcal{C}_n,\mathcal{D}_n), \quad (71)$$

with all the forms of $W(\theta;\mathcal{C},\mathcal{D})$ being the same as that of $W(\theta;\mathcal{A},\mathcal{B})$. Similarly we define the super potential $W(\theta;\mathcal{E},\mathcal{F})$ as above.

Then, $V^{\pm}(\theta;\mathcal{A},\mathcal{B})$ are defined as

$$\begin{aligned} V^{\pm}(\theta;\mathcal{A},\mathcal{B}) &= W^2(\theta;\mathcal{A},\mathcal{B}) \pm W'(\theta;\mathcal{A},\mathcal{B}) \\ &= \sum_{n=0}^{\infty} \beta^n V_n^{\pm}(\theta;\mathcal{A}_n,\mathcal{B}_n), \quad (72) \\ V^{\pm}(\theta;\mathcal{C},\mathcal{D}) &= W^2(\theta;\mathcal{C},\mathcal{D}) \pm W'(\theta;\mathcal{C},\mathcal{D}) \\ &= \sum_{n=0}^{\infty} \beta^n V_n^{\pm}(\theta;\mathcal{C}_n,\mathcal{D}_n), \quad (73) \\ V^{\pm}(\theta;\mathcal{E},\mathcal{F}) &= W^2(\theta;\mathcal{E},\mathcal{F}) \pm W'(\theta;\mathcal{E},\mathcal{F}) \\ &= \sum_{n=0}^{\infty} \beta^n V_n^{\pm}(\theta;\mathcal{E}_n,\mathcal{F}_n) \quad (74) \end{aligned}$$

The shape-invariance properties require for all $n \geq 0$

$$V_n^{+}(\theta;\mathcal{A}_n,\mathcal{B}_n) = V_n^{-}(\theta;\mathcal{C}_n,\mathcal{D}_n) + R_{n;m}(\mathcal{A}_n,\mathcal{B}_n), \quad (75)$$

$$V_n^{+}(\theta;\mathcal{C}_n,\mathcal{D}_n) = V_n^{-}(\theta;\mathcal{E}_n,\mathcal{F}_n) + R_{n;m}(\mathcal{C}_n,\mathcal{D}_n) \quad (76)$$

with $R_{n;m}(\mathcal{A}_n,\mathcal{B}_n), R_{n;m}(\mathcal{C}_n,\mathcal{D}_n)$ pure quantities. All parameters $\mathcal{C},\mathcal{D},\mathcal{E},\mathcal{F}$ can be derived from the parameters $\mathcal{A},\mathcal{B}$, which all are equal to one.

B. The recurrence relations obtained from relations (46)-(48)

In order to extend the formula (60) to SWSHs with $\beta \neq 0$, one applies Eqs.(46)-(48) to obtain

$$C_{0,0} = \frac{-2sB_{0,0}+1}{2m+1}, \quad (77)$$

$$D_{0,0} = -\frac{(2m+1)A_{0,0}+1}{2s} \quad (78)$$

$$E_{0,0} = \frac{-2sD_{0,0}+1}{2m+1}, \quad (79)$$

$$F_{0,0} = -\frac{(2m+1)C_{0,0}+1}{2s}, \quad (80)$$

which also can be derived from Eqs.(75)-(76) under the condition $n = 0$. When $n = 1$, Eqs.(75)-(76) gives

$$D_{1,1} = \frac{(2m+1)A_{0,0}-1}{(2m+1)C_{0,0}+1} B_{1,1}, \quad (81)$$

$$F_{1,1} = \frac{(2m+1)C_{0,0}-1}{(2m+1)E_{0,0}+1} D_{1,1}, \quad (82)$$

which become

$$D_{1,1} = -\frac{(2m+1)A_{0,0}-1}{2sB_{0,0}+2} B_{1,1}, \quad (83)$$

$$F_{1,1} = \frac{sB_{0,0}[(2m+1)A_{0,0}-1]}{[(2m+1)A_{0,0}+3][sB_{0,0}+1]} B_{1,1}. \quad (84)$$

For the general form of $n \geq 2$, similar calculation could proceed to give the corresponding $\mathcal{C}_n,\mathcal{D}_n$ and $\mathcal{E}_n,\mathcal{F}_n$, see appendix for details. Here we just show the results as following:

$$D_{n,p} = -2s\frac{D_{0,0}a_{n,p}}{\alpha_p b_{n,p}} C_{n,p} - \frac{U_{n,p}}{\alpha_p b_{n,p}} \quad (85)$$

$$\begin{aligned} C_{n,p-1} &= \frac{\left(\alpha_p - \frac{4s^2D_{0,0}^2}{\alpha_p}\right) a_{n,p}}{(\alpha_p - 1)a_{n,p-1}} C_{n,p} \\ &+ \frac{\check{U}_{n,p} - \frac{2sD_{0,0}}{\alpha_p} U_{n,p}}{(\alpha_p - 1)a_{n,p-1}}, \ p = 2,3,\ldots,[\frac{n+2}{2}], \quad (86) \end{aligned}$$

with $D_{n,[\frac{n+1}{2}]+1} = C_{n,[\frac{n}{2}]+1} = 0$, $\alpha_p = (2m+1)C_{0,0} + (2p-1)$ and $U_{n,p}, \check{U}_{n,p}$ being given by Eqs.(150)-(152) in the appendix.

Similarly, the quantities $E_{n,j}, F_{n,j}$ also can be calculated through $\mathcal{C}_n,\mathcal{D}_n$ by:

$$F_{n,p} = -2s\frac{F_{0,0}a_{n,p}}{\gamma_p b_{n,p}} E_{n,p} - \frac{Y_{n,p}}{\gamma_p b_{n,p}} \quad (87)$$

$$\begin{aligned} E_{n,p-1} &= 2s\frac{F_{0,0}b_{n,p}}{(\gamma_p-1)a_{n,p-1}} F_{n,p} + \frac{\gamma_p a_{n,p}}{(\gamma_p-1)a_{n,p-1}} E_{n,p} \\ &+ \frac{\check{Y}_{n,p}}{(\gamma_p-1)a_{n,p-1}} \quad (88) \end{aligned}$$

where $Y_{n,p}, \check{Y}_{n,p}, \gamma_p$ are given by Eqs.(154)-(156), (159) in the appendix.

The eigenfunctions ψ_n are obtained by the recurrence relation from Eq.(55):

$$\psi_n(\theta;\widetilde{\mathcal{C}},\widetilde{\mathcal{D}}) = \mathcal{A}^{-}(\theta;\mathcal{A},\mathcal{B})\psi_n(\theta;\mathcal{A},\mathcal{B}), \quad (89)$$

$$\psi_n(\theta;\mathcal{E},\mathcal{F}) = \mathcal{A}^{-}(\theta;\mathcal{C},\mathcal{D})\psi_n(\theta;\widetilde{\mathcal{C}},\widetilde{\mathcal{D}}) \quad (90)$$

$$\mathcal{A}^{-}(\theta;\mathcal{A},\mathcal{B}) = \frac{d}{d\theta} + W(\theta;\mathcal{A},\mathcal{B}) \quad (91)$$

$$\mathcal{A}^{-}(\theta;\mathcal{C},\mathcal{D}) = \frac{d}{d\theta} + W(\theta;\mathcal{C},\mathcal{D}). \quad (92)$$

Finally, through transferring eigenfunctions ψ_n into the spin-weighted spheroidal harmonics S_n by means of Eq.(29), that is

$$S_n(\theta;\mathcal{A},\mathcal{B}) = \frac{\psi_n(\theta;\mathcal{A},\mathcal{B})}{\sqrt{sin\theta}}, \quad (93)$$

$$S_n(\theta;\widetilde{\mathcal{C}},\widetilde{\mathcal{D}}) = \frac{\psi_n(\theta;\widetilde{\mathcal{C}},\widetilde{\mathcal{D}})}{\sqrt{sin\theta}}, \quad (94)$$

$$S_n(\theta;\mathcal{E},\mathcal{F}) = \frac{\psi_n(\theta;\mathcal{E},\mathcal{F})}{\sqrt{sin\theta}}, \quad (95)$$

we will rewrite S_n by the tradition as $S_n(\theta,m,s)$, that is,

$$\begin{aligned} S_n(\theta;\mathcal{A},\mathcal{B}) &= S_n(\theta,m,s), \quad (96) \\ S_n(\theta;\widetilde{\mathcal{C}},\widetilde{\mathcal{D}}) &= S_n(\theta,m+1,s), \quad (97) \\ S_n(\theta;\mathcal{E},\mathcal{F}) &= S_n(\theta,m+1,s-1). \quad (98) \end{aligned}$$

So Eq.(90) becomes

$$
\begin{aligned}
& S_n(\theta, m+1, s-1) \\
= & [\frac{d}{d\theta} + \frac{\cos\theta}{2\sin\theta} + W(\theta;\mathcal{C},\mathcal{D})]S_n(\theta, m+1, s) \\
= & \left[\frac{d}{d\theta} + \frac{s\cos\theta + (m+1)}{\sin\theta} + \beta\frac{ms}{(m+1)(s+1)}\sin\theta \right. \\
+ & \left. \sum_{n=2}^{\infty}\beta^n W_n(\theta;\mathcal{C}_n,\mathcal{D}_n)\right] S_n(\theta, m+1, s). \qquad (99)
\end{aligned}
$$

C. The recurrence relations obtained from relations (49)-(51)

We could proceed as before to obtain the extension of recurrence relations (3) for the spin-weighted spheroidal harmonics from Eqs. (49)-(51), which tell us

$$C_{0,0} = \frac{2sB_{0,0}+1}{2m+1}, \qquad (100)$$

$$D_{0,0} = \frac{(2m+1)A_{0,0}+1}{2s}, \qquad (101)$$

$$E_{0,0} = \frac{2sD_{0,0}+1}{2m+1}, \qquad (102)$$

$$F_{0,0} = \frac{(2m+1)C_{0,0}+1}{2s}. \qquad (103)$$

Eqs.(75)-(76) under the condition $n=1$ give the same formula as that of Eqs.(82)

$$D_{1,1} = \frac{(2m+1)A_{0,0}-1}{(2m+1)C_{0,0}+1}B_{1,1}, \qquad (104)$$

$$F_{1,1} = \frac{(2m+1)C_{0,0}-1}{(2m+1)E_{0,0}+1}D_{1,1}. \qquad (105)$$

So $D_{1,1},\ F_{1,1}$ turn out as

$$
\begin{aligned}
D_{1,1} &= \frac{(2m+1)C_{0,0}-1}{2sD_{0,0}+2}D_{1,1} \\
&= \frac{(2m+1)A_{0,0}-1}{2sB_{0,0}+2}B_{1,1}, \qquad (106)
\end{aligned}
$$

$$F_{1,1} = \frac{sB_{0,0}[(2m+1)A_{0,0}-1]}{[(2m+1)A_{0,0}+3][sB_{0,0}+1]}B_{1,1}. \qquad (107)$$

$D_{1,1}$ is different from Eqs.(84). As the formula Eqs.(81)-(82) for $D_{1,1},\ F_{1,1}$ are the same as Eqs.(105) , so we can use the formulas in the last subsection to obtain the subsequent $\mathcal{C}_n,\ \mathcal{D}_n,\ \mathcal{E}_n,\ \mathcal{F}_n$ for $n \geq 2$ as Eqs.(85)-(88). Note that actually the dependence of $\mathcal{C}_n,\ \mathcal{D}_n,\ \mathcal{E}_n,\ \mathcal{F}_n$ on the parameters $\mathcal{A},\ \mathcal{B}$ is different from that in the last subsection. With Eqs.(96)-(97) and the definition of

$$S_n(\theta, m+1, s+1) = S_n(\theta;\mathcal{E},\mathcal{F}) = \frac{\psi_n(\theta;\mathcal{E},\mathcal{F})}{\sqrt{sin\theta}}, \qquad (108)$$

the recurrence relations (90) now become

$$
\begin{aligned}
& S_n(\theta, m+1, s+1) \\
= & [\frac{d}{d\theta} + \frac{\cos\theta}{2\sin\theta} + W(\theta;\mathcal{C},\mathcal{D})]S_n(\theta, m+1, s) \qquad (109) \\
= & \left[\frac{d}{d\theta} - \frac{s\cos\theta + (m+1)}{\sin\theta} - \beta\frac{ms}{(m+1)(s+1)}\sin\theta \right. \\
+ & \left. \sum_{n=2}^{\infty}\beta^n W_n(\theta;\mathcal{C}_n,\mathcal{D}_n)\right] S_n(\theta, m+1, s). \qquad (110)
\end{aligned}
$$

The above equations are just the extension of formula (3) to the spin-weighted spheroidal harmonics.

VI. THE RECURRENCE RELATIONS FOR THE SWSHS UNDER THE CONDITION OF $\beta \neq 0$ II

In study of SWSHs with the method of SYSUQM, there are two kind forms for the super-potential W. The first one is that of (31), Eq.(32) [23], and is applied thus far to obtain the recurrence relations for different SWSHs [21]-[28] and in the above section. The second one is the form [29]-[30]:

$$W = W_0 + \sum_{n=1}^{\infty}\beta^n W_n \qquad (111)$$

$$W_n(\theta) = \sin\theta\sum_{k=0}^{n-1}a_{n,k}\cos^k\theta. \qquad (112)$$

except for $a_{0,0}$, We will write $a_{i,j}$ for all i,j and admit $a_{i,j}=0$ whenever one of the conditions $i \geq 1$ and $0 \leq j \leq i-1$ is violated. This will simplifying the calculation involved later. Please note that these parameters $a_{n,k}$ in the two forms of the super-potentials generally represent different quantities except for the case $n \leq 1$. For the same of simplicity, we do not denote new forms to them. The actually quantities of $a_{n,k}$ are given in Ref.[30]. The current section will provide the recurrence relations similar to Eqs.(99), (110) in the second form of the super-potential. Let $\mathcal{A}_n$ denote the set $B_{0,0}, A_{i,j}, i \leq n$, and $\mathcal{C}_n$ the set $D_{0,0}, C_{i,j}, i \leq n$ and $\mathcal{E}_n$ the set $F_{0,0}, E_{i,j}, i \leq n$. Similarly $\mathcal{A},\mathcal{C}$ represent the same physical contents as in the above section

$$W(\theta;\mathcal{A}) = W_0(A_{0,0},B_{0,0}) + \sum_{n=1}^{\infty}\beta^n W_n(\theta,\mathcal{A}_n) \qquad (113)$$

$$W_n(\theta;\mathcal{A}_n) = \sin\theta\sum_{k=0}^{n-1}a_{n,k}\mathcal{A}_{n,k}\cos^k\theta, \qquad (114)$$

and

$$W(\theta;\mathcal{C}) = W_0(C_{0,0},D_{0,0}) + \sum_{n=1}^{\infty}\beta^n W_n(\theta,\mathcal{C}_n) \qquad (115)$$

$$W_n(\theta;\mathcal{C}_n) = \sin\theta\sum_{k=0}^{n-1}a_{n,k}\mathcal{C}_{n,k}\cos^k\theta. \qquad (116)$$

and

$$W(\theta;\mathcal{E}) = W_0(E_{0,0},F_{0,0}) + \sum_{n=1}^{\infty}\beta^n W_n(\theta,\mathcal{E}_n) \quad (117)$$

$$W_n(\theta;\mathcal{E}_n) = \sin\theta\sum_{k=0}^{n-1} a_{n,k}\mathcal{E}_{n,k}\cos^k\theta. \quad (118)$$

the partner potentials related with the three super potentials are

$$V^{\pm}(\theta;\mathcal{A}) = W^2(\theta;\mathcal{A}) \pm W'(\theta;\mathcal{A}) = \sum_{n=0}^{\infty}\beta^n V_n^{\pm}(\theta;\mathcal{A}), \quad (119)$$

$$V^{\pm}(\theta;\mathcal{C}) = W^2(\theta;\mathcal{C}) \pm W'(\theta;\mathcal{C}) = \sum_{n=0}^{\infty}\beta^n V_n^{\pm}(\theta;\mathcal{C}), \quad (120)$$

$$V^{\pm}(\theta;\mathcal{E}) = W^2(\theta;\mathcal{E}) \pm W'(\theta;\mathcal{E}) = \sum_{n=0}^{\infty}\beta^n V_n^{\pm}(\theta;\mathcal{E}) \quad (121)$$

and the shape-invariance properties require the following to be met, that is

$$V_n^{+}(\theta;\mathcal{A}_n) = V_n^{-}(\theta;\mathcal{C}_n) + R_{n;m}(\mathcal{A}_n), \quad (122)$$

$$V_n^{+}(\theta;\mathcal{C}_n) = V_n^{-}(\theta;\mathcal{E}_n) + R_{n;m}(\mathcal{C}_n) \quad (123)$$

with $R_{n;m}(\mathcal{A}_n), R_{n;m}(\mathcal{C}_n)$ pure quantities. All parameters $\mathcal{C},\mathcal{E}$ can be derived from the parameters $\mathcal{A}$, which all are equal to one. Here we give the following formulae for $V_n^{\pm}(\theta;\mathcal{A}_n), n \geq 2$ for later use.

$$\begin{aligned} V_n^{\pm}(\theta;\mathcal{A}_n) &= -2\bar{b}_{0,0}\bar{a}_{n,0} \pm \bar{a}_{n,1} + \sum_{k=0}^{n-1}\bar{a}_{n-k,0}\bar{a}_{k,0} \\ &+ \sum_{k=0}^{n-1}\Big[\sum_{l=0}^{n-1}\sum_{i=0}^{n-1}(\bar{a}_{n-l,k+1-i} - \bar{a}_{n-l,k-i})\bar{a}_{l,i} \\ &\quad +2\bar{b}_{0,0}\bar{a}_{n,k+1} \mp (k+2)\bar{a}_{n,k+2} \\ &\quad +(2\bar{a}_{0,0} \pm (k+1))\bar{a}_{n,k}\Big]\cos^{k+1}\theta, \end{aligned} \quad (124)$$

where $\bar{b}_{0,0} = b_{0,0}B_{0,0}, \bar{a}_{n,k} = a_{n,k}A_{n,k}$. Similarly one could provide $V_n^{\pm}(\theta;\mathcal{C}_n), V_n^{\pm}(\theta;\mathcal{E}_n), n \geq 2$, which will be omitted.

In order to extend (2) to SWSHs, we first utilize the relations (46)-(48) to obtain the parameters $\mathcal{C},\mathcal{E}$ for fulfill the requirement of the shape-invariance properties of the super-potential. As stated before, $W_n, n \leq 1$ are the same in both Eqs.(31)-(32) and Eqs.(super-potential expansion2)-(112), so the results in subsection VA calculated from the shape-invariance requirements are valid for obtaining the parameters $\mathcal{C}_1$, and will be just rewritten here as

$$C_{0,0} = \frac{-2sB_{0,0}+1}{2m+1}, \quad (125)$$

$$D_{0,0} = -\frac{(2m+1)A_{0,0}+1}{2s} \quad (126)$$

$$E_{0,0} = \frac{-2sD_{0,0}+1}{2m+1}, \quad (127)$$

$$F_{0,0} = -\frac{(2m+1)C_{0,0}+1}{2s} \quad (128)$$

$$D_{1,1} = -\frac{(2m+1)A_{0,0}-1}{2sB_{0,0}+2}B_{1,1}, \quad (129)$$

$$F_{1,1} = \frac{sB_{0,0}[(2m+1)A_{0,0}-1]}{[(2m+1)A_{0,0}+3][sB_{0,0}+1]}B_{1,1}. \quad (130)$$

For $n \geq 2$, we can derive

$$\begin{aligned} C_{n,k} &= -\frac{2b_{0,0}D_{0,0}a_{n,k+1}C_{n,k+1} + (k+2)a_{n,k+2}C_{n,k+2}}{(2a_{0,0}C_{0,0} - (k+1)a_{n,k}} \\ &+ \frac{X_{n,k}}{2a_{0,0}C_{0,0} - (k+1)a_{n,k}} \end{aligned} \quad (131)$$

$$\begin{aligned} E_{n,k} &= -\frac{2b_{0,0}F_{0,0}a_{n,k+1}E_{n,k+1} + (k+2)a_{n,k+2}E_{n,k+2}}{(2a_{0,0}E_{0,0} - (k+1))a_{n,k}} \\ &+ \frac{\check{X}_{n,k}}{(2a_{0,0}E_{0,0} - (k+1))a_{n,k}} \end{aligned} \quad (132)$$

where $X_{n,k}, \check{X}_{n,k}$ are

$$\begin{aligned} X_{n,k} &= \sum_{i,l=0}^{n-1}\Big(a_{n-l,p}A_{n-l,p} - a_{n-l,p-1}A_{n-l,p-1}\Big)a_{l,i}A_{l,i} \\ &+ 2b_{0,0}B_{0,0}a_{n,k+1}A_{n,k+1} - (k+2)a_{n,k+2}A_{n,k+2} \\ &+ \Big(2a_{0,0}A_{0,0} + (k+1)\Big)a_{n,k}A_{n,k} \end{aligned} \quad (133)$$

and

$$\begin{aligned} \check{X}_{n,k} &= \sum_{i,l=0}^{n-1}\Big(a_{n-l,p}C_{n-l,p} - a_{n-l,p-1}C_{n-l,p-1}\Big)a_{l,i}C_{l,i} \\ &+ 2b_{0,0}D_{0,0}a_{n,k+1}C_{n,k+1} - (k+2)a_{n,k+2}C_{n,k+2} \\ &+ \Big(2a_{0,0}C_{0,0} + (k+1)\Big)a_{n,k}C_{n,k} \end{aligned} \quad (134)$$

where $p = k+1-i$. Therefore, the extended recurrence relations could be written as before

$$\begin{aligned} &S_n(\theta,m+1,s-1) \\ &= [\frac{d}{d\theta} + \frac{\cos\theta}{2\sin\theta} + W(\theta;\mathcal{C})]S_n(\theta,m+1,s) \\ &= \Big[\frac{d}{d\theta} + \frac{s\cos\theta+(m+1)}{\sin\theta} + \beta\frac{ms}{(m+1)(s+1)}\sin\theta \\ &+ \sum_{n=2}^{\infty}\beta^n W_n(\theta;\mathcal{C}_n)\Big]S_n(\theta,m+1,s). \end{aligned} \quad (135)$$

in the same way, we also extend (49)-51) as

$$
\begin{aligned}
& S_n(\theta, m+1, s+1) \\
= & \ [\frac{d}{d\theta} + \frac{\cos\theta}{2\sin\theta} + W(\theta; \mathcal{C})] S_n(\theta, m+1, s) \\
= & \left[\frac{d}{d\theta} - \frac{s\cos\theta + (m+1)}{\sin\theta} - \beta \frac{ms}{(m+1)(s+1)} \sin\theta \right. \\
+ & \left. \sum_{n=2}^{\infty} \beta^n W_n(\theta; \mathcal{C}_n)\right] S_n(\theta, m+1, s). \qquad (136)
\end{aligned}
$$

VII. DISCUSSION AND CONCLUSION

In summary, we have obtained the recurrence relations for the spin-weight spheroidal harmonics with the different spins, which are consistent in the case of $\beta = 0$ with the result given by R. Breuer et al. in Ref.[3]. Our methods apply SUSYQM to SWSHs, where the superpotential W is the key concept. By the sue of superpotential as in Eq.(32), we obtain the recurrence relations Eqs.(99), (110). Similarly, we give recurrence relations (135)-(136) through Eq.(114). Of course, these two kinds relations should be the same, as the first three terms shows. Whether using Eqs.(99), (110) or (135)-(136) depends on the form of SWSHs. By the methods of SUSYQM, we have investigated SWSHs thoroughly and the results will surely can be utilized numerically to obtain detailed informations on SWSHs.

However, our study give no information about the recurrence relations from half integer to integer spins of SWSHs. Further study should be how to extend this relations to that of SWSHs of spins being half integer and integer.

We give some examples about the present paper application. The results of Eqs.(99), (110) make one obtain the spin-weighted spheroidal harmonics $S_n(\theta, m, s-1)$ and $S_n(\theta, m, s+1)$ just from the spheroidal harmonics $S_n(\theta, m, s)|_{s=0}$. The spirit of this kind manufacturing process could also be used to study the radial Teukolsky equation. Thus, one will obtain the properties of the perturbation field ψ about $s = 1$ and $s = 2$ through the scalar perturbation field by use of recurrence relations, and this will give us new insight for the Kerr black hole perturbation study. Also the recurrence relations provide some information about the normalization constants concerning SWSHs, as it has already been done in Ref.[31]. Further extension of the study in the paper might be the application of the methods to study the radial Teukolsky equations, which might provide a new view to the stable problem of the Kerr black hole.

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VIII. APPENDIX: DETAILED CALCULATION

For the general form of $n \geq 0$, one simplifies the expressions of $V_n^{\pm}(\mathcal{A}_n, \mathcal{B}_n)$. With the help of

$$
W^2(\theta; \mathcal{A}_n, \mathcal{B}_n) = W_0^2 + \sum_{n=1}^{\infty} \beta^n W_n(\theta; \mathcal{A}_n, \mathcal{B}_n) + \sum_{n=2}^{\infty} \beta^n \sum_{k=1}^{n-1} W_k(\theta; \mathcal{A}_k, \mathcal{B}_k) W_{n-k}(\theta; \mathcal{A}_{n-k}, \mathcal{B}_{n-k}), \qquad (137)
$$

$$
W'(\theta; \mathcal{A}_n, \mathcal{B}_n) = W_0'(\theta; A_{0,0}, B_{0,0}) + \sum_{n=1}^{\infty} \beta^n W_n'(\theta; \mathcal{A}_n, \mathcal{B}_n), \qquad (138)
$$

by the use of Eq.(32), we obtain the formulae for $V_n^{\pm}$ in the case $n \geq 1$ as following

$$
\begin{aligned}
V_n^{\pm}(\theta; \mathcal{A}_n, \mathcal{B}_n) \ = \ & \cos\theta \sum_{p=1}^{[\frac{n+1}{2}]} \left[P_{n,p}^{\pm}(\mathcal{A}_n, \mathcal{B}_n) + G_{n,p}(A_{n-1}, B_{n-1})\right] \sin^{2p-2}\theta \\
& + \sum_{p=1}^{[\frac{n}{2}]+1} \left[Q_{n,p}^{\pm}(\mathcal{A}_n, \mathcal{B}_n) + H_{n,p}(A_{n-1}, B_{n-1})\right] \sin^{2p-2}\theta \qquad (139)
\end{aligned}
$$

where

$$
\begin{aligned}
P^{\pm}_{n,p}(\mathcal{A}_n,\mathcal{B}_n) &= 2b_{0,0}B_{0,0}A_{n,p}a_{n,p} + \Big(2a_{0,0}A_{0,0} \pm (2p-1)\Big)B_{n,p}b_{n,p}, && (140)\\
Q^{\pm}_{n,p}(\mathcal{A}_n,\mathcal{B}_n) &= 2b_{0,0}B_{0,0}B_{n,p}b_{n,p} + 2a_{0,0}A_{0,0}a_{n,p}A_{n,p} - 2a_{0,0}A_{0,0}a_{n,p-1}A_{n,p-1}\\
&\quad \pm(2p-1)A_{n,p}a_{n,p} \mp (2p-2)A_{n,p-1}a_{n,p-1} && (141)\\
G_{n,p}(\mathcal{A}_{n-1},\mathcal{B}_{n-1}) &= \sum_{k=1}^{n-1}\sum_{j=1}^{[\frac{n}{2}]+1}\Big[b_{k,p-j}B_{k,p-j}a_{n-k,j}A_{n-k,j} + a_{k,p-j}A_{k,p-j}b_{n-k,j}B_{n-k,j}\Big], && (142)\\
H_{n,p}(\mathcal{A}_{n-1},\mathcal{B}_{n-1}) &= \sum_{k=1}^{n-1}\sum_{j=1}^{[\frac{n}{2}]+1}\Big[b_{k,p-j}B_{k,p-j}b_{n-k,j}B_{n-k,j}\\
&\quad + a_{k,p-j}A_{k,p-j}a_{n-k,j}A_{n-k,j} - a_{k,p-1-j}A_{k,p-1-j}a_{n-k,j}A_{n-k,j}\Big], && (143)
\end{aligned}
$$

where $a_{n,j}=0$ whenever $j<0$ or $j>[\frac{n}{2}]$ and $b_{n,j}=0$ whenever$j<0$ or $j>[\frac{n+1}{2}]$. Similarly, we have

$$
\begin{aligned}
V^{\pm}_n(\theta;\mathcal{C}_n,\mathcal{D}_n) &= \cos\theta\sum_{p=1}^{[\frac{n+1}{2}]}\Big[P^{\pm}_{n,p}(\mathcal{C}_n,\mathcal{D}_n) + G_{n,p}(C_{n-1},D_{n-1})\Big]\sin^{2p-2}\theta\\
&\quad + \sum_{p=1}^{[\frac{n}{2}]+1}\Big[Q^{\pm}_{n,p}(\mathcal{C}_n,\mathcal{D}_n) + H_{n,p}(C_{n-1},D_{n-1})\Big]\sin^{2p-2}\theta && (144)
\end{aligned}
$$

where

$$
\begin{aligned}
P^{\pm}_{n,p}(\mathcal{C}_n,\mathcal{D}_n) &= 2b_{0,0}D_{0,0}C_{n,p}a_{n,p} + \Big(2a_{0,0}C_{0,0} \pm (2p-1)\Big)D_{n,p}b_{n,p}, && (145)\\
Q^{\pm}_{n,p}(\mathcal{C}_n,\mathcal{D}_n) &= 2b_{0,0}D_{0,0}b_{n,p}B_{n,p} + 2a_{0,0}C_{0,0}a_{n,p}C_{n,p} - 2a_{0,0}C_{0,0}a_{n,p-1}C_{n,p-1}\\
&\quad \pm(2p-1)C_{n,p}a_{n,p} \mp (2p-2)C_{n,p-1}a_{n,p-1} && (146)\\
G_{n,p}(\mathcal{C}_{n-1},\mathcal{D}_{n-1}) &= \sum_{k=1}^{n-1}\sum_{j=1}^{[\frac{n}{2}]+1}\Big[b_{k,p-j}D_{k,p-j}a_{n-k,j}C_{n-k,j} + a_{k,p-j}C_{k,p-j}b_{n-k,j}D_{n-k,j}\Big], && (147)\\
H_{n,p}(\mathcal{C}_{n-1},\mathcal{D}_{n-1}) &= \sum_{k=1}^{n-1}\sum_{j=1}^{[\frac{n}{2}]+1}\Big[b_{k,p-j}D_{k,p-j}b_{n-k,j}D_{n-k,j}\\
&\quad + a_{k,p-j}C_{k,p-j}a_{n-k,j}C_{n-k,j} - a_{k,p-1-j}C_{k,p-1-j}a_{n-k,j}C_{n-k,j}\Big], && (148)
\end{aligned}
$$

where $a_{n,j}=0$ whenever $j<0$ or $j>[\frac{n}{2}]$ and $b_{n,j}=0$ whenever$j<0$ or $j>[\frac{n+1}{2}]$.

One could use the shape-invariance equation Eq.(75) to obtain

$$
\begin{aligned}
P^{-}_{n,p}(\mathcal{C}_n,\mathcal{D}_n) &= P^{+}_{n,p}(\mathcal{A}_n,\mathcal{B}_n) + G_{n,p}(\mathcal{A}_{n-1},\mathcal{B}_{n-1}) - G_{n,p}(\mathcal{C}_{n-1},\mathcal{D}_{n-1}) && (149)\\
&\equiv U_{n,p}, && (150)\\
Q^{-}_{n,p}(\mathcal{C}_n,\mathcal{D}_n) &= H_{n,p}(\mathcal{A}_{n-1},\mathcal{B}_{n-1}) + Q^{+}_{n,p}(\mathcal{A}_n,\mathcal{B}_n) - H_{n,p}(\mathcal{C}_{n-1},\mathcal{D}_{n-1}) && (151)\\
&\equiv \check{U}_{n,p}, && (152)
\end{aligned}
$$

and from Eq.(145)and Eq.(146) we can get

$$
\begin{aligned}
P^{-}_{n,p}(\mathcal{C}_n,\mathcal{D}_n) &= -\alpha_p D_{n,p}b_{n,p} - D_{0,0}C_{n,p}a_{n,p}\\
Q^{-}_{n,p}(\mathcal{C}_n,\mathcal{D}_n) &= -\alpha_p C_{n,p}a_{n,p} - D_{0,0}D_{n,p}b_{n,p} + (\alpha_p-1)C_{n,p-1}a_{n,p-1},
\end{aligned}
$$

where $\alpha_p=(2m+1)C_{0,0}+(2p-1)$. In the same way, we have

$$
\begin{aligned}
Y_{n,p} &= P^{+}_{n,p}(\mathcal{C}_n,\mathcal{D}_n) + G_{n,p}(\mathcal{C}_{n-1},\mathcal{D}_{n-1}) - G_{n,p}(\mathcal{E}_{n-1},\mathcal{F}_{n-1}) && (153)\\
&= P^{-}_{n,p}(\mathcal{E}_n,\mathcal{F}_n) && (154)\\
\check{Y}_{n,p} &= H_{n,p}(\mathcal{C}_{n-1},\mathcal{D}_{n-1}) + Q^{+}_{n,p}(\mathcal{C}_n,\mathcal{D}_n) - H_{n,p}(\mathcal{E}_{n-1},\mathcal{F}_{n-1}) && (155)\\
&= Q^{-}_{n,p}(E_{n,j},F_{n,j}), && (156)
\end{aligned}
$$

where

$$\begin{aligned} P^-_{n,p}(\mathcal{E}_n, \mathcal{F}_n) &= -\gamma_p E_{n,p} a_{n,p} - 2sF_{0,0} F_{n,p} b_{n,p} && (157) \\ Q^-_{n,p}(E_{n,j}, F_{n,j}) &= -\gamma_p E_{n,p} a_{n,p} - 2sF_{0,0} F_{n,p} b_{n,p} + (\gamma_p - 1) E_{n,p-1} a_{n,p-1}, && (158) \\ \gamma_p &= (2m+1) E_{0,0} + (2p-1). && (159) \end{aligned}$$

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